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Short Communication

Identification of a crack in a beam based on the finite element method of a B-spline wavelet on the interval

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Abstract

The model-based forward and inverse problems in the diagnosis of structural crack location and size by using the finite element method of a B-spline wavelet on the interval (FEM BSWI) were studied. First the crack and uncracked elements of BSWI were built to solve the forward problem. The first three frequencies influencing functions of normalized crack location and size are approximated by means of surface-fitting techniques. Then the first three measured natural frequencies are employed as inputs of the functions. The intersection of the three frequencies contour lines predicted the normalized crack location and size. Both the numerical and experimental studies verified the validity of the BSWI elements in solving crack singular problems with high performance.

1. Introduction

Vibration-based methods, irrespective of whether the basis is the mode shape or frequency, have so far been intended for exploitation for the detection of structural cracks [1–6]. Nandwana and Maiti [1] modeled the crack as a rotational spring and give a semi-analytical solution for beams. Meanwhile, finite element method (FEM) is employed for identification of a crack in structures due to the fact that FEM is firmly established as a standard procedure for the solution of crack problems when the crack is represented by a rotational spring (or other crack model) of stiffness or flexibility. Kisa [2] integrated the FEM and component mode synthesis for a cracked Timoshenko beam. Lee [3] used the lowest four natural frequencies and cracked FEM model to detect a structural crack. Lele and Maiti [4], and Patil and Maiti [5], respectively, employed eight-node iso-parametric elements to make a more efficient calculation for single and multiple cracks identification in beams. Unlike traditional FEM, Daubechies wavelet-based finite element method (WFEM) was employed for modal analysis of crack problems with good performance [6]. The desirable advantages of WFEM are multi-resolution properties and various basis functions for structural analysis. By means of "two-scale relations" of scaling functions, the scale adopted can be changed freely according to requirements to improve analysis accuracy. However, due to Daubechies wavelets lacking of the explicit function expression, traditional numerical

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integrals such as Gaussian integrals cannot provide desirable precision, and one it needs to calculate connection coefficients [7], which is a complex process.

Because B-spline wavelet on the interval (BSWI) scaling functions have explicit expressions, the element stiffness and mass matrices can be calculated conveniently. Furthermore, B-spline wavelets have the best approximation properties among all known wavelets of a given order L [8]. The application of the BSWI basis to the versatile FEM provides accurate analytical results and a robust multi-level solving process [9,10]. So BSWI element is used to identify crack location and size in a beam using the first three measured frequencies.

2. Modal analysis based on BSWI element

The free vibration frequency equations for a multi-degree of freedom (mdofs) system are

$$\left|\overline{\mathbf{K}} - \omega^2\overline{\mathbf{M}}\right| = 0 \tag{1}$$

where $\overline{\mathbf{K}}$ and $\overline{\mathbf{M}}$ are the global stiffness and mass matrices that can be obtained from the standard assembly procedure of elemental stiffness matrix $\mathbf{K}_b^{e,j}$and elemental consistent mass matrix $\mathbf{M}_b^{e,j}$.

The elemental stiffness matrix $\mathbf{K}_b^{e,j}$ can be solved by

$$\mathbf{K}_b^{e,j} = \frac{EI}{l_e^3}\int_0^1 \mathbf{R}_b^{\mathrm{T}}\left(\frac{\mathrm{d}^2\boldsymbol{\phi}}{\mathrm{d}\xi^2}\right)\left(\frac{\mathrm{d}^2\boldsymbol{\phi}^{\mathrm{T}}}{\mathrm{d}\xi^2}\right)\mathbf{R}_b\,\mathrm{d}\xi, \tag{2}$$

where EI is the bending stiffness, $\boldsymbol{\phi} = \left\{\phi^j_{m,-m+1}(\xi)\phi^j_{m,-m+2}(\xi)\ldots\phi^j_{m,2^j-1}(\xi)\right\}^{\mathrm{T}}$ is the column vector combined by the BSWI scaling functions for order m at the scale j on the interval [0, 1] (the explicit expression of the functions can be seen in Refs. [10,11]), l_e is the elemental length and the transformation matrix $\mathbf{R}_b$ [7,11] is given by

$$\mathbf{R}_b = \left(\left[\boldsymbol{\phi}(\xi_1), \frac{1}{l_e}\frac{\mathrm{d}\boldsymbol{\phi}(\xi_1)}{\mathrm{d}\xi}, \boldsymbol{\phi}(\xi_2)\ldots\boldsymbol{\phi}(\xi_{r-1}), \boldsymbol{\phi}(\xi_r), \frac{1}{l_e}\frac{\mathrm{d}\boldsymbol{\phi}(\xi_r)}{\mathrm{d}\xi}\right]^{\mathrm{T}}\right)^{-1}, \tag{3}$$

where r is the elemental nodal number.

The elemental consistent mass matrix $\mathbf{M}_b^{e,j}$ is given by

$$\mathbf{M}_b^{e,j} = l_e\rho A\int_0^1 \mathbf{R}_b^{\mathrm{T}}\boldsymbol{\phi}(\boldsymbol{\phi})^{\mathrm{T}}\mathbf{R}_b\,\mathrm{d}\xi \tag{4}$$

where ρ is the density and A is the area of the cross section. To deal with the boundary conditions, the corresponding dofs are set to zero and eliminated from the equations.

3. Crack identification method using BSWI element

3.1. Forward problem

Because the natural frequencies can be easily and cheaply acquired in practice and the linear rotational spring model can effectively describe open crack, the present work is based on the open crack in structures and using the first three natural frequencies to identify crack location and size. The physical and rotational spring models are shown in Fig. 1.

The continuity conditions at crack position indicate that the left node j and right node $j+1$ have the same transverse displacement, namely, $u_j = u_{j+1}$, while their rotations θ_j and θ_{j+1} are connected through the cracked stiffness submatrix $\mathbf{K}_S$.

$$\mathbf{K}_S = \begin{bmatrix} K_t & -K_t \\ -K_t & K_t \end{bmatrix}, \tag{5}$$

where K_t [1] is defined by

$$K_t = bh^2E/(72\pi\alpha^2 f(\alpha)), \tag{6}$$

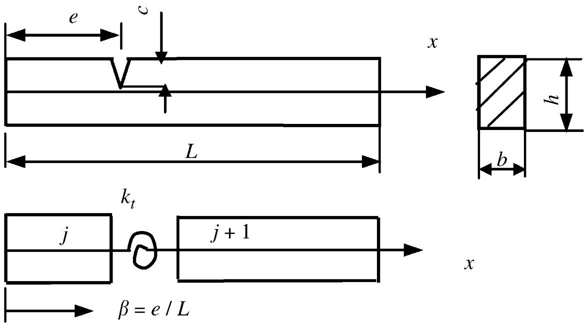


Fig. 1. Cracked beam and rotational spring model.

in which

$$f(\alpha) = 0.6384 - 1.035\alpha + 3.7201\alpha^2 - 5.1773\alpha^3 + 7.553\alpha^4 - 7.332\alpha^5 + 2.4909\alpha^6, \tag{7}$$

where $\alpha = c/h$ denotes normalized crack size and $\beta = e/L$ in Fig. 1 denotes normalized crack location.

Hence, we can assemble the cracked stiffness submatrix $\mathbf{K}_S$ into the global stiffness matrix easily. The global mass matrix of the cracked structure is equal to the uncracked one. Then, the cracked structural finite element model of BSWI is constructed. The solution of the eigenvalue problem can then proceed as usual.

For the determination of the natural frequencies ω for a given crack location (determine the location of the cracked stiffness submatrix in global stiffness) and size (determine K_t), the normalized crack location β and size α are given as inputs. The relationship between the natural frequencies and the crack parameters (normalized crack location and size) is

$$\omega_j = F_j(\alpha, \beta) \quad (j = 1, 2, 3, \ldots). \tag{8}$$

Because the functions F_j $(j = 1, 2, 3, \ldots)$ are unknown and the discrete values can be acquired through Eq. (1), the surface-fitting techniques [12] are needed for three-dimensional plots of Eq. (8).

Example. Taking the simple supported beam for example, beam length $L = 0.5$ m, Young's modulus $E = 2.1 \times 10^{11}$ Pa, $h \times b = 0.02\,\text{m} \times 0.012\,\text{m}$, Poisson's ratio $\mu = 0.3$ and $\rho = 7860\,\text{kg/m}^3$. We adopt the scaling functions of BSWI for $m = 4$ at the scale $j = 3$ to construct elements. Fig. 2 shows the relationship between ω_i $(i = 1, 2, 3)$ and all possible crack parameters using surface-fitting techniques (here, $\beta \in [0.05, 0.9]$, $\alpha \in [0.05, 0.7]$).

3.2. Inverse problem

The crack identification inverse problem can be described by

$$(\alpha, \beta) = F_j^{-1}(\omega_j) \quad (j = 1, 2, 3, \ldots). \tag{9}$$

The measurements of any two natural frequencies enable us to define the normalized location and size of a crack if Eq. (8) has been determined. However, when we use the crack identification method of frequency contour plots [12], two natural frequency contour plots may intersect at more than one point. Therefore a minimum of three frequencies is required to identify the two unknown parameters of the normalized crack location and size. Because the first three frequencies can be measured easily and accurately, they are usually served as inputs to solve the inverse problem in structure damage identification.

If the first three frequencies are known, the frequency contour plots of Eq. (8) can be acquired and plotted on the same axes. The common intersection of all the three contour lines indicates the normalized crack location and size. This intersection becomes unique due to the fact that any cracked structural natural frequency can be represented by a frequency equation (see Eq. (8)) that is dependent on normalized crack parameters [12].

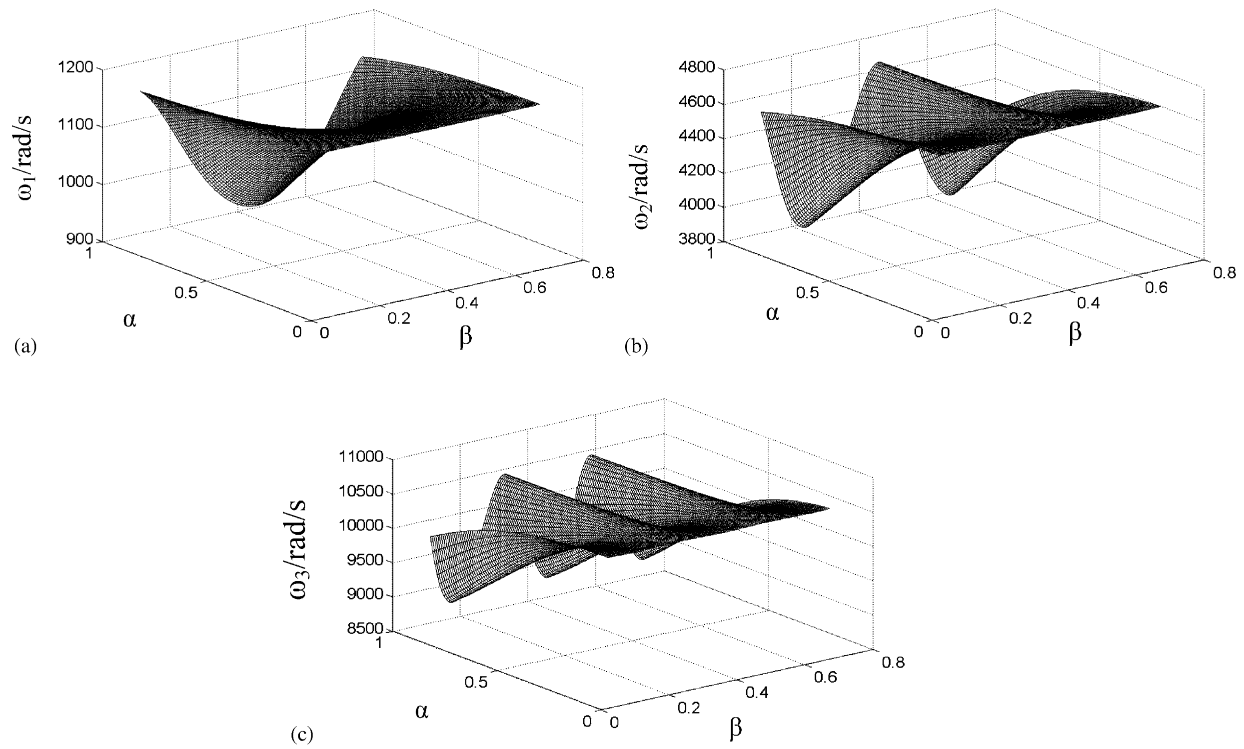
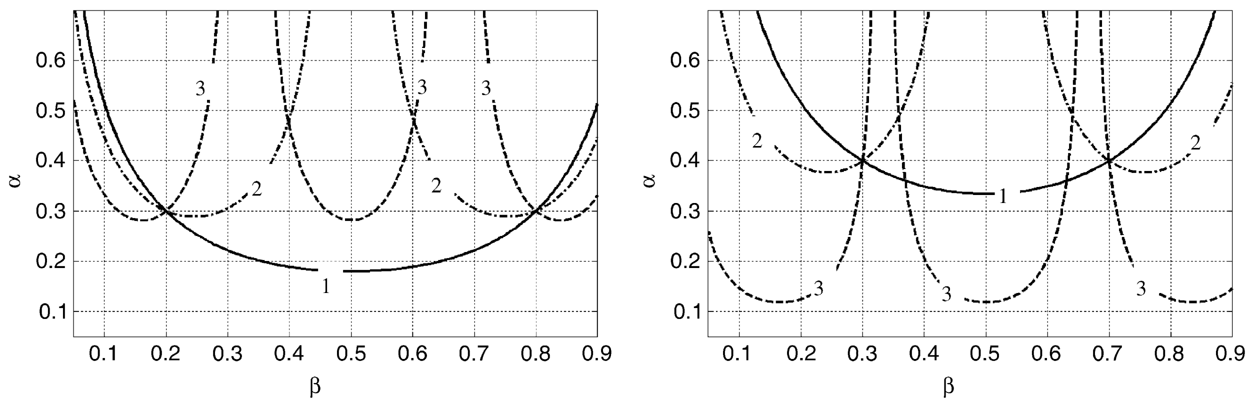


Fig. 2. Relationship between frequencies and all possible crack parameters.

Table 1
Comparison of predicted and actual crack parameters

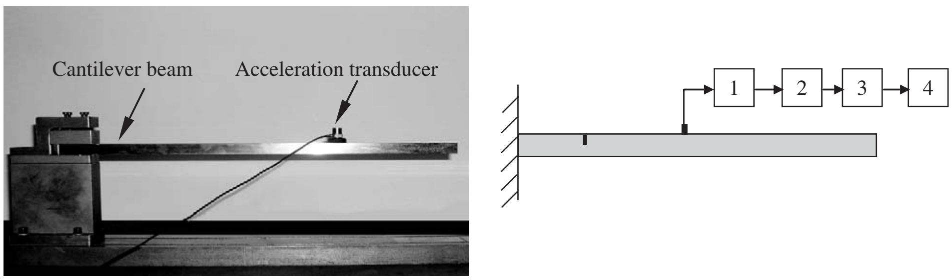
Case	Actual	Actual	Exact natural frequencies (rad/s)			Predicted β^*	Predicted α^*
	β	α	ω_1	ω_2	ω_3	(error %)	(error %)
1	0.1	0.1	1177.66	4705.604	10573.663	0.1001(0.10)	0.1001(0.10)
2	0.2	0.1	1176.402	4694.446	10562.889	0.1999(0.05)	0.1(0)
3	0.2	0.3	1162.672	4557.815	10281.642	0.1999(0.05)	0.3001(0.03)
4	0.3	0.2	1165.559	4644.925	10587.306	0.3001(0.03)	0.2(0)
5	0.3	0.3	1149.363	4563.174	10568.267	0.3001(0.03)	0.3(0)
6	0.4	0.2	1160.876	4686.538	10544.768	0.4(0)	0.2(0)
7	0.4	0.4	1105.78	4608.053	10369.596	0.4(0)	0.4(0)
8	0.5	0.2	1159.108	4712.566	10436.419	0.501(0.2)	0.2(0)
9	0.5	0.4	1099.58	4712.566	9964.053	0.501(0.2)	0.4(0)
10	0.6	0.6	1003.482	4479.635	10090.524	0.6(0)	0.6(0)
11	0.7	0.4	1124.204	4446.84	10541.653	0.6999(0.014)	0.4001(0.025)
12	0.7	0.6	1042.566	4140.357	10474.076	0.7001(0.014)	0.6001(0.017)
13	0.8	0.6	1100.028	4060.228	9522.737	0.8001(0.038)	0.6001(0.017)

Continuing to *Example* in Section 3.1, the inverse problem is solved by using the exact first three frequencies [1] as inputs. The comparison of predicted and actual crack parameters is shown in Table 1. The predicted normalized crack location and size have very perfect solving precision. The relative errors are not more than 0.2%. Fig. 3 shows the crack identification results of some cases by using the contour lines.



Case 3 Case 11

Fig. 3. Frequency contour plots of Case 3 and Case 11. 1. 1st frequency; 2. 2nd frequency; 3. 3rd frequency.



(a) Experimental set (b) measuring principle chart

Fig. 4. Experimental set measuring principle chart: (a) experimental set; (b) measuring principle chart. 1. Charge amplifier; 2. PXI signal conditioner; 3. data acquisition card; 4. computer.

4. Experimental verification

In order to verify the validity of the structural crack identification methods, the experimental set of a cantilever beam is built (shown in Fig. 4(a)). The measuring principle chart is shown in Fig. 4(b).

The material of workpiece for the experiment is 45# steel, the cantilever beam length $L = 0515\,\text{m}$, the cross section $h \times b = 0.02\,\text{m} \times 0.012\,\text{m}$, Young's modulus $E = 2.06 \times 10^{11}$ Pa, material density $\rho = 7917\,\text{kg/m}^3$, Poisson's ratio $\mu = 0.3$ and kerfs width is 0.02 mm. Crack cases are shown in Table 3. For the simple structure, single input and single output (SISO) modal analysis by using a hammer as excitation is a usually used method.

In most cases, however, the three lines do not intersect at one point because of inaccuracies in the modeling as compared to measured results. For this purpose, the 'zero-setting' procedure described by Adams [13] is used. In this procedure, Young's modulus of the structure is changed by using the undamaged natural frequencies of the structure to determine an effective value, and is given by the following iterative approach:

$$\left| \omega_i^2 \overline{\mathbf{M}} - E_m \frac{\overline{\mathbf{K}}}{E} \right| = 0, \tag{10}$$

where E_m is the corrected value of Young's modulus E, which can be acquired through solving Eq. (10) for each frequency. The measured frequencies of the cracked cantilever beam and the values of corrected E_m are shown in Table 2.

In this section, the first three experimental measured frequencies are employed as inputs of the inverse problem for crack quantitative identification. Fig. 5 shows the crack identification results in a cantilever beam

Table 2
Measured frequencies of cracked cantilever beam and the values of corrected E_m

Case	β	α	ω_1	E_m	ω_2	E_m	ω_3	E_m
Uncracked			358.1	1.73346E11	2324.8	1.85999E11	6483.0	1.84450E11
1	0.6	0.2	355.9		2289.0		6350.3	
2	0.6	0.4	355.6		2241.5		6216.7	
3	0.8	0.2	357.8		2309.4		6301.2	
4	0.8	0.4	357.7		2251.1		6180.8	

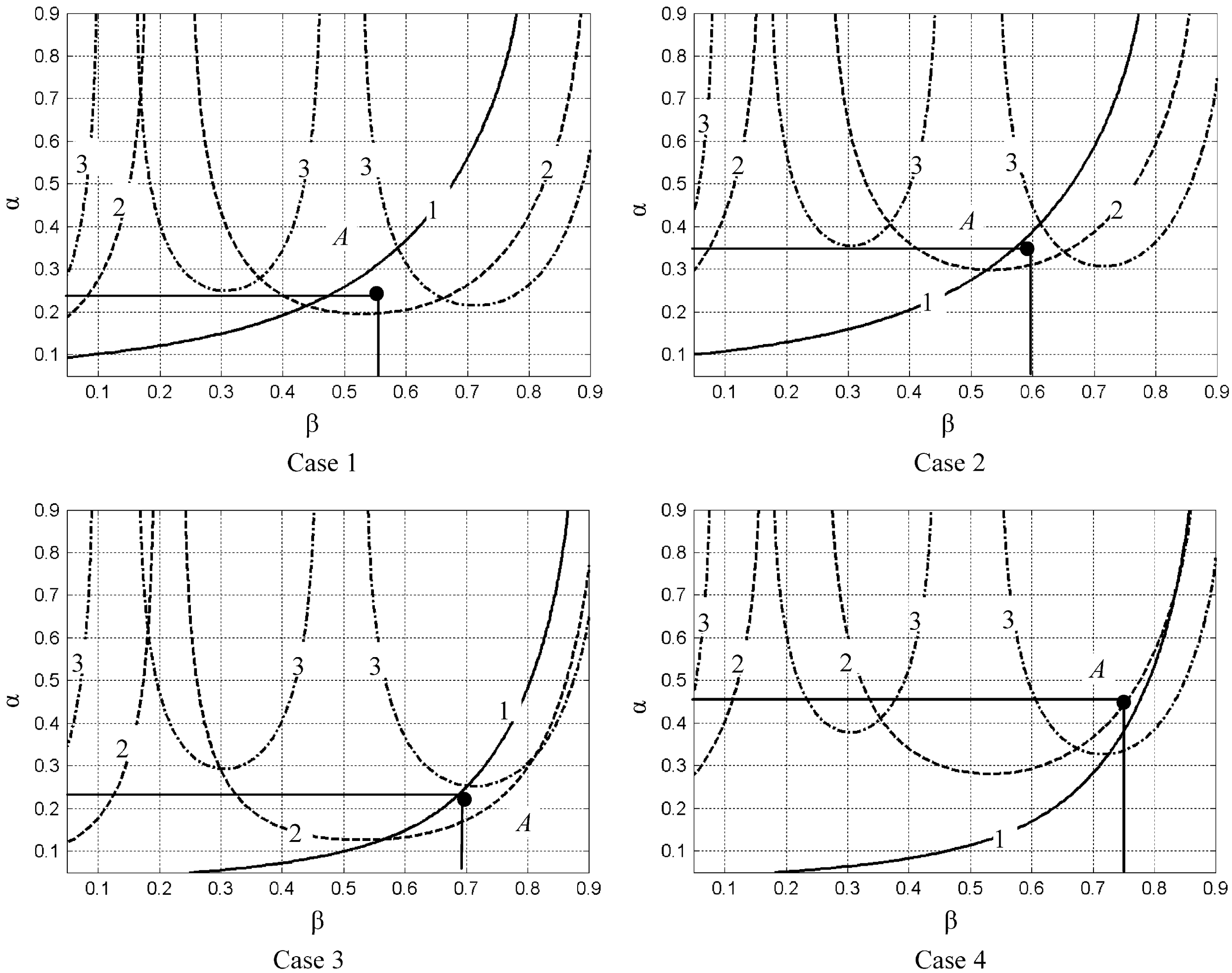


Fig. 5. Frequency contour plots of Cases 1–4. 1. 1st frequency; 2. 2nd. frequency; 3. 3rd frequency.

using the frequency contour plots. The intersection point A of three lines indicates the normalized crack location and size. In the experimental studies, when the three lines do not meet exactly, the centroid of the three pairs of intersections is taken as the normalized crack location and size [1]. Table 3 shows the comparison of actual crack parameters and the predicted crack parameters. The relative errors of the given cases are not more than 18%.

Table 3
Comparision of actual crack locations and predicted results

Case	β	α	β^* (error %)	α^* (error %)
1	0.6	0.2	0.562(6.3)	0.225(12.5)
2	0.6	0.4	0.596(0.7)	0.349(12.8)
3	0.8	0.2	0.693(13.4)	0.235(17.5)
4	0.8	0.4	0.745(6.9)	0.464(16.0)

5. Conclusions

A methodology based on BSWI element to detect crack location and size is presented. Because of the good character of BSWI scaling functions, the BSWI element presented in this paper is a useful tool to deal with high-performance computation in structural crack identification. Numerical and experimental investigations verify that the proposed method can be utilized to detect crack location as well as crack size with high performance.

Acknowledgements

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